Exercise 1

1. An urn contains five balls numbered 1 to 5 of which the first three are silver and the last two are gold. A sample of size two is drawn with replacement. Let E_1 denote the event that the first ball drawn is silver, and E_2 denote the event that the second ball drawn is silver.

   (a) Describe the sample space for the experiment, and sketch the events E_1, E_2 and $E_1 \cap E_2$

   (b) Find $P(E_1)$, $P(E_2)$ and $P(E_1 \cap E_2)$.

   (c) Repeat (a) and (b) for sampling without replacement.

2. A garage has twelve motors to sell, two of which are faulty. The seller can crate the motors with all twelve in one box or with six in two boxes. The customer will inspect two of the twelve motors if they are crated in one box, and one motor from each of the two boxes if they are crated six to a box. The garage has three possible strategies:

   (a) Crate all twelve in one box;

   (b) Put one faulty motor in each of two boxes of six.

   (c) Put the two faulty motors in one of the two boxes of six.

   What is the probability that the customer will not inspect either of the faulty motors under each of the three strategies?

3. The game of "odd one out" is played by three people, each flipping a single coin. All three flip their coins simultaneously; if one face is different from the other two, its owner loses the game.

   (a) What is the probability that there is a loser on a given turn, assuming all coins are fair?

   (b) If there is no loser on the first turn, the coins are all flipped again until there is a loser. What is the probability that an even number of turns is required to determine who loses the game?

4. Person A spins an fair coin three times and B spins it twice. By considering all possible results, show that A has probability $\frac{1}{2}$ of getting more heads than B. Show that the same is true if A spins the coin n+1 times and B n times.

5. If 6 people, among whom are A and B, stand in a row, what is the probability that there will be exactly r people between A and B? If they stand in a ring, show that the probability of exactly r people between A and B in the clockwise direction is 1/5. Try replacing 6 by n for general problems of the same sort.

6. An urn contains 2 red disks lettered A and B, 2 green disks lettered A and B, and 4 blue disks lettered A, B, C and D. One disk is drawn and its colour and letter noted. Are the following events independent?

   A, red; (A or D), red; (A or blue), red.

7. Suppose that the genders of all children in a family are independent and that boys and girls are equally probable, that is, both have probability 0.5.

   (a) For families of three children calculate the probabilities of the events A, B and $A \cap B$ where $A =$ 'There are children of both genders', $B =$ 'Not more than one child is a girl'.

   (b) Do the same for families of four children.

   Are A and B independent events in 2(a) and 2(b)?

8. A, B, and C are events of positive probability. For each of the following either supply a proof or a counter-example.

   (a) If A and B are independent, A and B^c are independent.

   (b) If $P(A \mid B) < P(A)$, then $P(B \mid A) < P(B)$.

   (c) If $P(A \cap B \mid C) = P(A \mid C)P(B \mid C)$ then A and B are independent.

   (d) If $P(B \mid A) = P(B \mid A^c)$ then A and B are independent.

9. An fair die is thrown n times. Find a formula for the probability that each face appears at least once. [Hint: let A_i be the event that the ith face does not appear and expand $P(A_1 \cup \ldots \cup A_6)$.]

10. Let X and Y be mutually exclusive events with known probabilities $P(X)$ and $P(Y)$. Give an expression for $P(X \cup Y)$. By partitioning appropriate sets into disjoint subsets and using just the result for $P(X \cup Y)$, carefully prove;

    (a) $P(A^c) = 1 - P(A)$,

    (b) $P(A^c \cap B) = P(B) - P(A \cap B)$,

    (c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,

    (d) $P((A \cup B) \cap (A^c \cup B^c)) = P(A) + P(B) - 2P(A \cap B)$.

    How would you interpret the event $(A \cup B) \cap (A^c \cup B^c)$?

11. In a colony of exceedingly rare sea worms, the individuals are either red or green.

    (a) Among the population, 10% of the worms are vicious. One fifth of vicious worms are green and one quarter of non-vicious worms are red. While diving we see a red worm; what is the probability that it is vicious?

    (b) A worm's colour is partly determined by the colour of its ancestors (the worms reproduce by budding). The conditional probabilities are $\frac{3}{4}$, $\frac{2}{3}$ and $\frac{4}{5}$ conditioning respectively on the parent being red, the grandparent being red, and both the parent and grandparent being red. A worm is chosen at random.

       i. What is the probability that the worm is red and has a red parent?

       ii. What is the probability that the worm is red or its parent is red or its grandparent is red?

12. One urn contains four red disks and five green, and a second contains five red and four green. One disk is transferred from the first to the second urn and a disk is then withdrawn from the second urn. Find the probability that it is red. What is the probability of a red disk if two disks are transferred rather than one?

Exercise 2

1. Find the distribution functions corresponding to the following density functions:

(i)	$f_X(x) = 1/[\pi(1+x^2)]$	$-\infty < x < \infty$	(Cauchy)
(ii)	$f_X(x) = e^{-x}/(1+e^{-x})^2$	$-\infty < x < \infty$	(Logistic)
(iii)	$f_X(x) = (a-1)/(1+x)^a$	$0 < x < \infty$	(Pareto)
(iv)	$f_X(x) = c\tau x^{\tau-1}e^{-cx^\tau}$	$0 < x < \infty$, $\tau > 0$, $c > 0$	(Weibull).

2. Find (without generating functions) the mean and the variance for the following distributions

   (a) $f_X(x) = \begin{cases} e^{-kx}x^{r-1}k^r/(r-1)! & x \geq 0 \\ 0 & x < 0 \end{cases}$ $\quad r > 0$, $k > 0$

   (The Gamma Distribution)

   (b) $f_X(x) = e^{-\lambda}\lambda^x/x!$ $\quad x = 0, 1, 2, \ldots.$ $\lambda > 0$

   (Poisson Distribution)

   (c) $f_X(x) = \binom{a+x-1}{x}\left[\frac{b}{1+b}\right]^a\left[\frac{1}{1+b}\right]^x$ $\quad x = 0, 1, 2, \ldots.$ $a > 0$, $b > 0$

   (Negative Binomial)

   (d) $f_X(x) = \frac{a-1}{(1+x)^a}$ $\quad x > 0$, $a > 1$

   (Pareto)

3. Show that if a random variable X is normally distributed with density, $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$ for $-\infty < x < \infty$, then $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

4. Suppose that X is a continuous random variable taking values between $-\infty$ and ∞ with distribution function $F(x)$. Sometimes we want to *fold* the distribution of X about the value $x = a$, that is we want the distribution function $F_Y(y)$ of the random variable Y obtained from X by taking $Y = X - a$ if $X > a$ and $Y = a - X$ if $X < a$. Find $F_Y(y)$ by working out directly $P(Y \leq y)$. What is the density function of Y? A particularly important application is the case when X has a $N(\mu, \sigma^2)$ distribution, and $a = \mu$. Apply your result to this case.

5. Show that if X is a continuous positive random variable

$$E[X] = \int_0^\infty [1 - F_X(x)]dx \ .$$

   (Try integration by parts.)

6. If X is a continuous positive continuous variable with density function $f_X(x)$ and mean μ, show that

$$f(y) = \begin{cases} 0 & y < 0 \\ yf_X(y)/\mu & y \geq 0 \end{cases}$$

   is a density function, and hence show that

$$\mathrm{E}(X^3)\mathrm{E}(X) \geq \{\mathrm{E}(X^2)\}^2.$$

7. A random variable has 'no memory' if for all x and for $y > 0$

$$P[X > x + y \mid X > x] = P[X > y].$$

Show that if X has either the exponential distribution, or a geometric distribution with $\text{Prob}(X = x) = q^{x-1}p$, then X has no memory. Interpret this property.

8. In a simple queueing system, the time spent in the queue has the following distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - \rho \exp(-\lambda(1-\rho)x) & x \geq 0 \end{cases}$$

where $\lambda \geq 0$ and $0 < \rho < 1$. Check that this is a distribution function, and describe the form of the distribution in words.

9. If X is a nonnegative continuous random variable, then

$$\bar{F}(x) = P(X > x)$$

is called the *survival function*, and

$$f(x)/\bar{F}(x)$$

is called the *failure rate* at x.

(a) Find the failure rate for the Weibull and Pareto distributions.

(b) Show that in general the failure rate does not decrease as x increases if

$$\bar{F}(x+y)/\bar{F}(x)$$

does not increase as x increases, for all $y \geq 0$. (Differentiate the logarithm of the given expression with respect to x.)

Exercise 3

1. Find the moment generating function of $N(\mu, \sigma^2)$. Find the mean and the variance of $Y = e^X$ if $X \sim N(\mu, \sigma^2)$. (Y has the Lognormal distribution, very popular as a skew distribution for positive variables.)

2. State and prove the Markov Inequality. Use it to establish that, if Y is a random variable with MGF $M_Y(t)$, then for $\gamma > 0$ and $t > 0$,
$$P(Y \geq \gamma) \leq \frac{M_Y(t)}{e^{\gamma t}}.$$
Hence show that if Z has a standard normal distribution, then for $\gamma \geq 0$,
$$P(Z \geq \gamma) \leq e^{-\gamma^2/2}.$$

3. (a) Let X be a random variable with MGF $M_X(t)$. Find an expression for the cumulant generating function $K_X(t)$ as a function of the moments m_r. Hence find expressions for κ_2 and κ_3 in terms of these moments.
   (b) Let $Y = X - m_1$. Find expressions for κ_2, κ_3, κ_4 and κ_5 as functions of the moments of Y.
   (c) Now let $Z = Y/\sigma$. Find an expression for $K_Z(t)$ as a function of $K_Y(t)$. Hence find κ_2, κ_3 and κ_4 for Z as functions of the moments of Y.
   (d) Finally, suppose that X is distributed $N(\mu, \sigma^2)$. Write down $K_X(t)$ in this case. Use your results in (a) to find m_2 and m_3 for X.

4. Find the moment generating function of the Double Exponential or Laplace distribution with density function
$$f_X(x) = \frac{1}{2}e^{-|x|} \quad -\infty < x < \infty,$$
and hence its first four cumulants. (It has been suggested that the Laplace distribution is more typical of naturally occurring physical measurements than the normal distribution.)

5. If $H(x)$ and $G(y)$ are distribution functions, for which of the following definitions is $F(x, y)$ a joint distribution function? (Sometimes one can give a simple description of the joint distribution of X and Y which leads to the form given. The notations $\max(a, b)$ and $\min(a, b)$ mean the largest of a, b and the smallest of a, b respectively.)
   (a) $F(x, y) = H(x) + G(y)$
   (b) $F(x, y) = H(x)G(y)$ (use independence)
   (c) $F(x, y) = \max[H(x), G(y)]$
   (d) $F(x, y) = \min[H(x), H(y)]$ (consider $X = Y$).

Exercise 4

1. The probability that an insect lays r eggs is Poisson with mean λ, and the probability that an egg develops is p. Assuming independence between the eggs, find the probability that k eggs survive, $k = 0, 1, 2, \ldots$. (We have a joint distribution for (K, R) and want the marginal probability function for K.)

2. Suppose that the number of insurance claims that a particular policyholder makes in one year has a Poisson distribution with mean Λ, and that over the large population of policyholders Λ has a Gamma distribution. Find the probability of x claims in one year $x = 0$, 1,... by a policyholder chosen at random from the population of policyholders. (There is a joint distribution for (X, Λ), and we require the marginal distribution of X.)

3. Suppose that a bus is X minutes late, where X has an exponential distribution with mean $1/\Theta$, where Θ varies randomly with the density of the traffic according to the Gamma density function

$$f(\theta) = \lambda^\alpha \theta^{\alpha-1} e^{-\lambda\theta} / \Gamma(\alpha).$$

   What is the density function of X over all values of Θ? (Write down the joint density function for (X, Θ) and integrate out over values of Θ to get the marginal density function of X.)

4. If X and Y have joint distribution given by

$$f_{X,Y}(x, y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   find the covariance of X and Y, and the conditional density function of Y given $X = x$.

5. Let the joint density function for X, Y be

$$f_{X,Y}(x, y) = \begin{cases} 8xy & 0 < y < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find $E[Y|X = x]$.
   (b) Find $\text{Var}[Y|X = x]$.
   (c) Find $E[XY|X = x]$, and hence the covariance of X and Y.

6. Suppose that the joint density function of (X, Y) is

$$f_{X,Y}(x, y) = \begin{cases} 1 - a(1 - 2x)(1 - 2y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   where $-1 < a < 1$.

   Prove that X and Y are independent if and only if X and Y are uncorrelated.

7. Find the density function of the sum of two independent exponential random variables with the same mean. Do the same for the case where the means are different.

8. Let

$$Y = \begin{cases} \sum_{i=1}^{N} X_i & \text{if } N \geq 1 \\ 0 & \text{if } N = 0 \end{cases}$$

where $\{X_i\}$, $i = 1, 2, \ldots$ is a sequence of iid random variables and N is a discrete random variable taking values $0, 1, \ldots$ independent of $\{X_i\}$. Let $K_Y(t), K_N(t), K_X(t)$ denote the cumulant generating functions of Y, N and X_i respectively. Prove that

$$K_Y(t) = K_N[K_X(t)].$$

Furthermore, suppose that $P(N = i) = p^{i-1}(1-p)$ for $i = 1, 2, \ldots$ and the density of X_i is given by $\alpha^2 x e^{-\alpha x}$ for $x > 0$. Prove that Y has density function

$$\frac{\alpha(1-p)}{2\sqrt{p}} \left(e^{-\alpha(1-\sqrt{p})y} - e^{-\alpha(1+\sqrt{p})y} \right),$$

for $y > 0$.

9. Suppose that X and Y are bivariate normally distributed with zero means, unit variances and correlation coefficient ρ. Show that the correlation coefficient between X^2 and Y^2 is ρ^2. (Use bivariate cumulants.)

10. Find the joint moment generating function of X and X^2 where X is a standard normal random variable.

11. Find the moment generating function for the product of two independent standard normal random variables, X and Y, by first finding the conditional moment generating function of XY for a fixed value of X. Hence or otherwise show that if U, V, X and Y are independent standard normal random variables, then $UV + XY$ has a Laplace distribution.

12. If X and Y are iid exponential random variables, find the joint density of $U = X/Y$ and $V = X+Y$.

13. Suppose that (R, Θ) are the polar coordinates of the random variables (X, Y) and that R has the density function $f_R(r)$ and is independent of Θ which has a uniform distribution on $(0, 2\pi)$. Find an expression for the joint density of (X, Y) in terms of f_R. Interpret your result geometrically.

14. If (X, Y) have some joint distribution, then one might try to define the concept of 'Y stochastically bigger than X' by the property $F_X(z) > F_Y(z)$ for all real z. Show that

    (a) If $F_X(z) > F_Y(z)$ for all z, then $E[Y] > E[X]$. You may assume X and Y are continuous random variables.

    (b) If $F_X(z) > F_Y(z)$ for all z, then $P[Y > X] > 0$.

    (c) If X and Y are independent continuous random variables, and $F_X(z) > F_Y(z)$ for all z, then $P[Y > X] > 0.5$.